Permutation Invariant Parking Assortments

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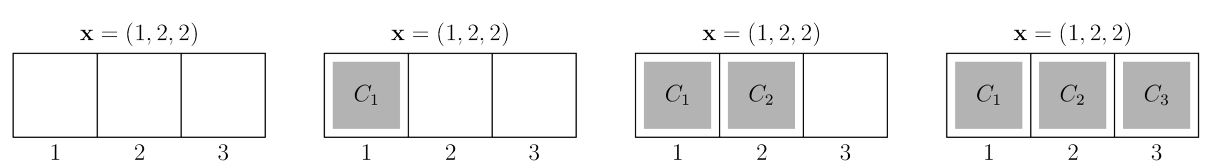
Introduction and Background

Definition (parking functions):

- Consider a one-way street with $n \in \mathbb{N}$ parking spots.
- There are n cars waiting to enter the street sequentially; each car has a parking spot preference.
- When a car enters the street, it attempts to park in its preference; if it is occupied, the car continues driving down the street until it finds an unoccupied spot in which to park (if there is one).
- If the cars' preferences allow them all to park, we say it is a *parking function of length n*.

Example: Let $\mathbf{x} = (1, 2, 2)$.



History:

- These structures were introduced by Konheim and Weiss in their study of hashing functions.
- For a fixed n, the number of parking functions is $(n+1)^{n-1}$; the proof is a classical argument that makes use of a "circular parking lot."

Definition (parking assortments):

- There are $n \in \mathbb{N}$ cars of assorted lengths $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ entering a one-way street containing $m = \sum_{i=1}^{n} y_i$ parking spots.
- The cars have parking preferences $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [m]^n$ and enter the street in order.
- For each i, car i starts looking for parking at its spot x_i and parks in the first y_i contiguously available spots thereafter (if there are any).
- If all cars are able to park under the preference list $\mathbf{x}$, we say that $\mathbf{x}$ is a *parking assortment* for $\mathbf{y}$.
- For a fixed $\mathbf{y}$, let $\mathsf{PA}_n(\mathbf{y})$ denote its set of parking assortments.

Note that any rearrangement of the entries of a parking function also results in a parking function. However, if $\mathbf{y} = (1, 2, 2)$, then $\mathbf{x} = (1, 2, 1)$ is a parking assortment, whereas its rearrangement $\mathbf{x}' = (2, 1, 1)$ is neither.

Definition (invariance): Given $\mathbf{y} \in \mathbb{N}^n$ and $\mathbf{x} \in \mathsf{PA}_n(\mathbf{y})$, we say $\mathbf{x}$ is an *invariant parking assortment for* $\mathbf{y}$ if all of the rearrangements of $\mathbf{x}$ are also in $\mathsf{PA}_n(\mathbf{y})$. Let $\mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$ denote the set of invariant parking assortments for $\mathbf{y}$ and $\mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y})$ denote the set of nondecreasing invariant parking assortments for $\mathbf{y}$.

Definition (degree and characteristic): Let $\mathbf{y} \in \mathbb{N}^n$. For any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$, the *degree* of $\mathbf{x}$ is given by

$$\deg \mathbf{x} := |\{i \in [n] : x_i \neq 1\}|.$$

Moreover, the *characteristic* of $\mathbf{y}$ is given by

$$\chi(\mathbf{y}) := \max_{\mathbf{z} \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})} \deg \mathbf{z}.$$

Notation (list operations): For $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$, $k \in \mathbb{N}$, and $i \in [n]$, let $\mathbf{v}_{\hat{i}} := (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \in \mathbb{N}^{n-1}$ and $\mathbf{v}_{|_i} := (v_1, v_2, \ldots, v_i) \in \mathbb{N}^i$.

Minimal Characteristic Results

Theorem (Chen, Harris, Martínez, Pabón, and Sargent 2022): Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. If $\chi(\mathbf{y}) = 0$, then $\chi(\mathbf{y}_{|_i}) = 0$ for all $i \in [n]$.

Theorem (Chen, Harris, Martínez, Pabón, and Sargent 2022): Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. Then, $\chi(\mathbf{y}) = 0$ if and only if there does not exist $w \in \mathbb{N}_{>1}$ such that $(1^{n-1}, w) \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$.

Remark: This is a concise characterization for $\mathbf{y} \in \mathbb{N}^n$ to have minimal degree in the sense that there are only n distinct permutations of $(1^{n-1}, w)$, so we only need to perform mn parking experiments.

Theorem (Chen, Harris, Martínez, Pabón, and Sargent 2022): Let $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{N}^3$. Then $\chi(\mathbf{y}) = 0$ if and only if $y_1 < y_2$, $y_1 < y_3$, and $y_1 + y_3 \neq y_2$.

Theorem (Chen 2023): Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. If $\chi(\mathbf{y}) = 0$, then

$$y_1 < \min(y_2, y_3, \ldots, y_n) \quad \text{and} \quad y_2 \neq \sum_{j \in [n] \setminus \{2\}} y_j.$$

Maximal Characteristic Results

Theorem (Chen 2023): Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. Then $\chi(\mathbf{y}) = n - 1$ if and only if

$$y_1 \geq y_2 \quad \text{and} \quad y_2 = y_3 = \cdots = y_n.$$

Structural and Enumerative Results

Theorem (Chen 2023): Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ and $\mathbf{y}^+ = (\mathbf{y}, y_{n+1}) \in \mathbb{N}^{n+1}$.

- If $(1^{n-d}, \mathbf{w}) \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$, where $\mathbf{w} \in \mathbb{N}_{>1}^d$, then $(1^{n-d+1}, \mathbf{w}_{\hat{i}}) \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$ for all $i \in [d]$.
- If $\mathbf{x} \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$, then $(1, \mathbf{x}) \in \mathsf{PA}_{n+1}^{\mathrm{inv}}(\mathbf{y}^+)$. In particular, we have the embedding

$$\eta : \begin{cases} \mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y}) & \hookrightarrow \mathsf{PA}_{n+1}^{\mathrm{inv},\uparrow}(\mathbf{y}^+) \\ \mathbf{x} & \mapsto (1, \mathbf{x}). \end{cases}$$

Theorem (Chen 2023): Let $\mathbf{y} = (b, a^{n-1}) \in \mathbb{N}^n$, where $n \geq 2$.

- If $a \mid b$ or $b > (n-1)a$, then $\mathbf{x} \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$ if and only if

$$x_{(i)} \in \{1 + (k-1)a : k \in [i]\} \quad \forall i \in [n].$$

  Moreover,

$$|\mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})| = (n+1)^{n-1} \quad \text{and} \quad |\mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y})| = \frac{1}{n+1}\binom{2n}{n}.$$

- Otherwise, if $a \nmid b$ and $b < (n-1)a$, then $\mathbf{x} \in \mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$ if and only if

$$x_{(i)} \in \begin{cases} \{1 + (k-1)a : k \in [i]\} & \forall i \in \left[\left\lfloor \frac{b}{a} \right\rfloor + 1\right] \\ \left\{1 + (k-1)a : k \in \left[\left\lfloor \frac{b}{a} \right\rfloor + 1\right]\right\} & \text{otherwise.} \end{cases}$$

  Moreover,

$$|\mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})| = \sum_{j=0}^{n - \lfloor b/a \rfloor - 1} (-1)^j \binom{n}{j} \left(n - \left\lfloor \frac{b}{a} \right\rfloor - 1\right)^j (n - j + 1)^{n-j-1} \quad \text{and}$$

$$|\mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y})| = \frac{n - \lfloor b/a \rfloor + 1}{n+1} \binom{n + \lfloor b/a \rfloor}{\lfloor b/a \rfloor}.$$

Remark: Some of these enumerative formulas are deduced via two results: an enumerative result related to the theory of the Pitman-Stanley polytope and empirical distributions and a recursive formula for Catalan's triangle.

Towards Computing $\mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})$

Theorem (Chen, Harris, Martínez, Pabón, and Sargent 2022): Let $\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2$.

- $y_1 < y_2 \implies \mathsf{PA}_2^{\mathrm{inv}}(\mathbf{y}) = \{(1, 1)\}$.
- $y_1 \geq y_2 \implies \mathsf{PA}_2^{\mathrm{inv}}(\mathbf{y}) = \{(1, 1), (1, y_2 + 1), (y_2 + 1, 1)\}$.

Theorem (Chen, Harris, Martínez, Pabón, and Sargent 2022): Let $a < b < c$ be in $\mathbb{N}$. Then the following table provides car lengths $\mathbf{y} \in \{a, b, c\}^3$ and the corresponding nondecreasing parking assortments.

$\mathbf{y}$	$\mathsf{PA}_3^{\mathrm{inv},\uparrow}(\mathbf{y})$
(a, a, a)	$(1,1,1), (1,1,1+a), (1,1,1+2a), (1,1+a,1+a), (1,1+a,1+2a)$
(a, a, b)	$(1,1,1), (1,1,1+a)$
$(a, b, a),\ b = 2a$	$(1,1,1), (1,1,1+a), (1,1,1+2a)$
$(a, b, a),\ b \neq 2a$	$(1,1,1), (1,1,1+a)$
$(b, a, a),\ 2a \leq b$	$(1,1,1), (1,1,1+a), (1,1,1+2a), (1,1+a,1+a), (1,1+a,1+2a)$
$(b, a, a),\ 2a > b$	$(1,1,1), (1,1,1+a), (1,1+a,1+a)$
(a, b, b)	$(1,1,1)$
(b, a, b)	$(1,1,1), (1,1,1+a)$
(b, b, a)	$(1,1,1), (1,1,1+a), (1,1,1+b), (1,1,1+a+b)$
(a, b, c)	$(1,1,1)$
$(a, c, b),\ a+b = c$	$(1,1,1), (1,1,1+a+b)$
$(a, c, b),\ a+b \neq c$	$(1,1,1)$
(b, a, c)	$(1,1,1), (1,1,1+a)$
$(b, c, a),\ a+b = c$	$(1,1,1), (1,1,1+a), (1,1,1+a+b)$
$(b, c, a),\ a+b \neq c$	$(1,1,1), (1,1,1+a)$
$(c, a, b),\ a+b \leq c$	$(1,1,1), (1,1,1+a), (1,1,1+a+b)$
$(c, a, b),\ a+b > c$	$(1,1,1), (1,1,1+a)$
$(c, b, a),\ a+b \leq c$	$(1,1,1), (1,1,1+a), (1,1,1+b), (1,1,1+a+b)$
$(c, b, a),\ a+b > c$	$(1,1,1), (1,1,1+a), (1,1,1+b)$

An Extremal Result

Theorem (Chen 2023): Let $\mathbf{y} \in \mathbb{N}^n$. Then

$$|\mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y})| \leq \binom{2^{n-1} + n - 2}{n-1}.$$

Open Problems

1. Let $\mathbf{y} \in \mathbb{N}^n$. Do $|\mathsf{PA}_n^{\mathrm{inv}}(\mathbf{y})| \leq (n+1)^{n-1}$ and $|\mathsf{PA}_n^{\mathrm{inv},\uparrow}(\mathbf{y})| \leq \frac{1}{n+1}\binom{2n}{n}$ hold for any n?
2. Let $\mathbf{y} \in \mathbb{N}^n$, where $\chi(\mathbf{y}) = \alpha$, and $\mathbf{y}^+ = (\mathbf{y}, y_{n+1}) \in \mathbb{N}^{n+1}$. What must be true about $\mathbf{y}$ and $\mathbf{y}^+$ to guarantee that $\chi(\mathbf{y}^+) = \alpha$?

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