A hyperexponential-like model for car waiting times at a left turn light without assumptions on opposing traffic

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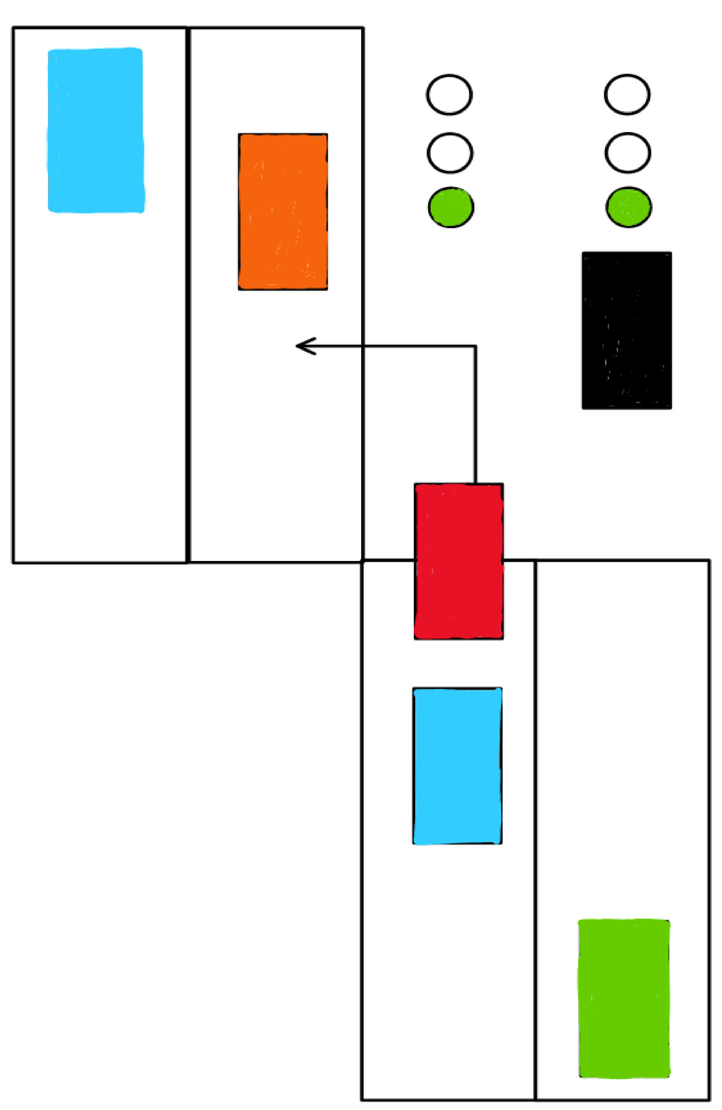
Abstract

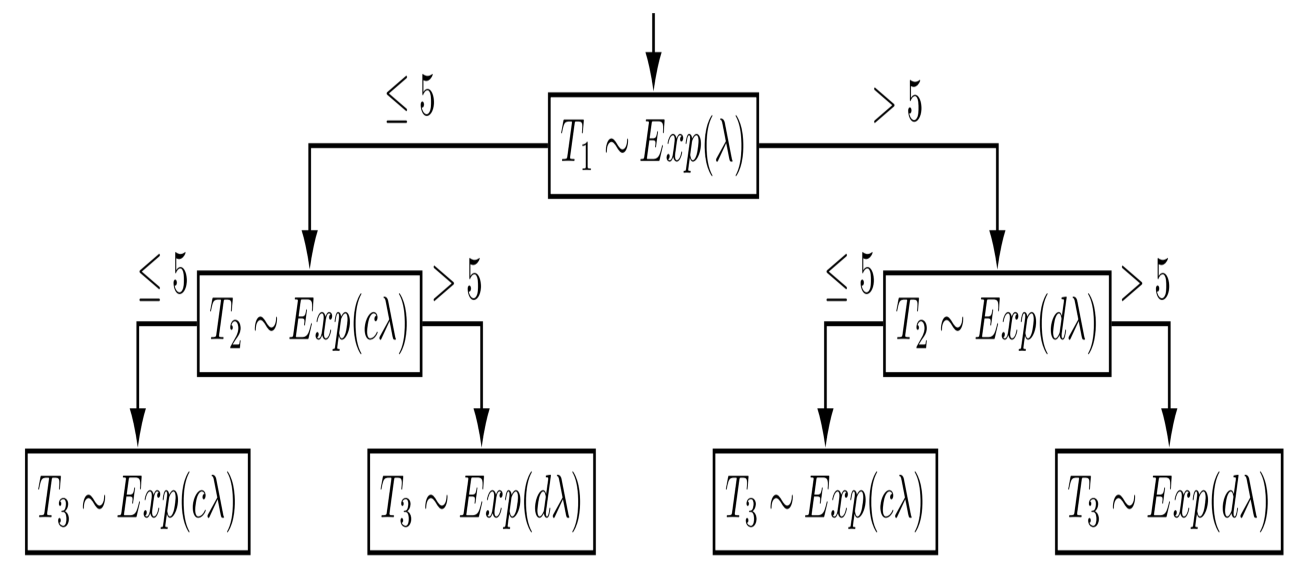
Understanding the traffic flow at intersections can have a wide array of benefits, including optimizing signals for faster commutes and reducing gas emissions. One scenario of interest is how to allow cars to make left turns at an intersection. One frequently used framework for this situation is by allowing cars to yield to the opposing traffic and make the left turn once there is enough space to do so. Most mathematical models to describe this scenario rely on making assumptions or having knowledge on the distribution of cars arriving at the intersection in the opposing lane(s). We propose a model that only assumes the distribution of the time it takes cars to make the left turn and uses the realized time it takes a car to make the left turn as a proxy for the behavior of the opposing traffic to influence the leaving time for the next car. This model incorporates the hyperexponential distribution, a mixture density of the exponential distribution, to allow for this flexibility. We show that this model has desirable asymptotic properties and examine through simulation the potential for the sum of car leave times to be approximately normally distributed.

Introduction



One commonly encountered scenario at a four-way intersection is when a car is intending to make a left turn, a maneuver that requires them to cross through the opposing lanes. Frequently, this situation is addressed by allowing cars to yield to opposing traffic and make the left turn when there is enough space to do so. There are a wide variety of mathematical models used in this situation to better understand the flow of traffic at these turns, but they commonly make assumptions about the flow of traffic in the opposing lane to understand the time it will take for cars to make the left turn. We also seek to understand how long it will take a given car to make the left turn upon arrival, but do not want to require any knowledge about the flow of opposing traffic. We propose a model where the influencing factor on the time it takes any given car to make the left turn is the amount of time it took the previous car to make the left turn.

Model Setup



- The time that the 1st car in line waits (T1) to make the left turn follows an exponential distribution with some rate parameter λ.
- If the 1st car makes the turn in less than 5 seconds, the wait time of the 2nd car (T2) follows an exponential distribution with parameter cλ.
- If the 1st car takes more than 5 seconds to make the turn, the parameter is dλ. This process repeats for each subsequent car.
- The time each car waits to make the turn upon being first in line follows a hyperexponential-like distribution; it has an Exp(cλ) distribution with some probability and an Exp(dλ) distribution with some probability.

Results

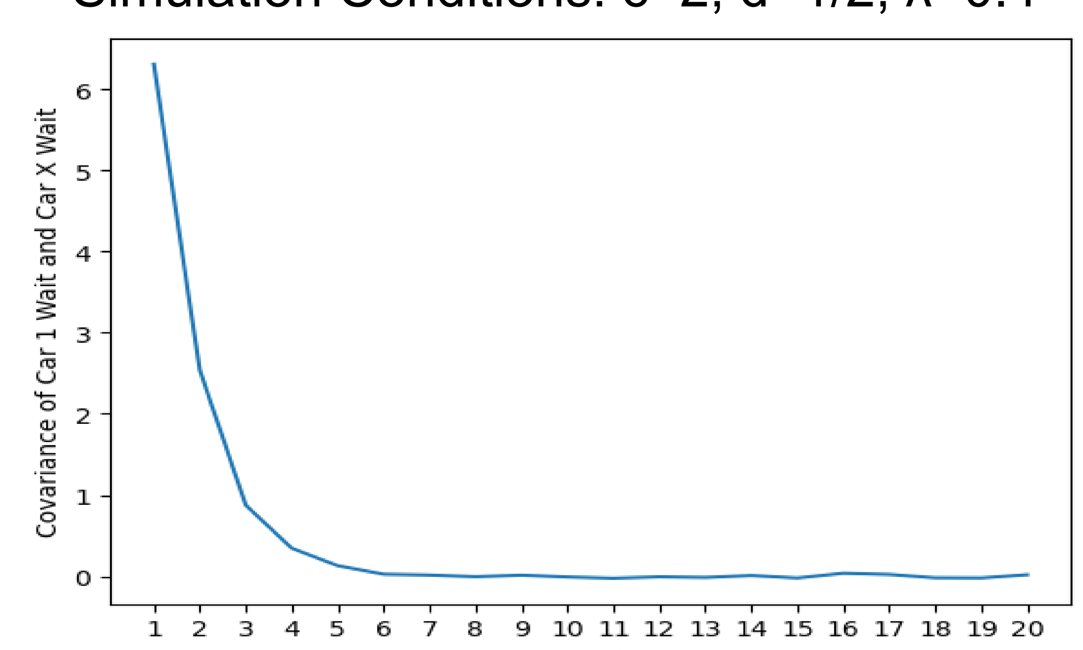
For $n \geq 2$, the distribution of T_n is a hyperexponential distribution: T_n is an exponential random variable and its rate parameter is given by $c\lambda$ with probability $1 - x_n$ and $d\lambda$ with probability x_n, where $\{x_n\}$ is a sequence defined as follows:

Define the function $f(x) = (1-x)e^{-5c\lambda} + xe^{-5d\lambda}$
Let $x_2 = e^{-5\lambda}$. For $n = 3, 4, ...,$ let $x_n = f(x_{n-1})$

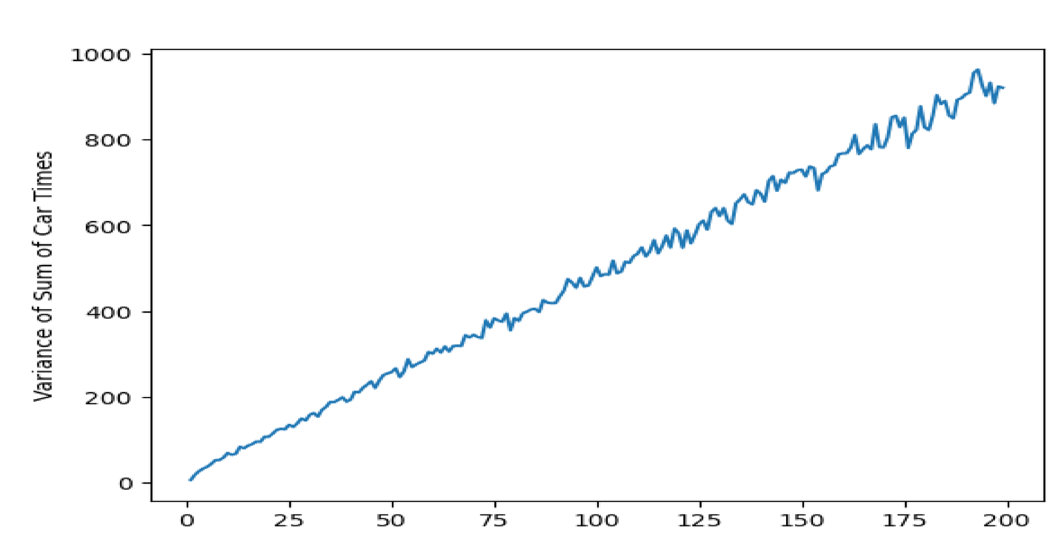
This leads to the following findings:

1. The sequence $\{x_n\}$ converges to $\frac{e^{-5c\lambda}}{1+e^{-5c\lambda}-e^{-5d\lambda}}$
2. For $n \geq 2$: $\mathbf{E}[T_n] = \frac{1-x_n}{c\lambda} + \frac{x_n}{d\lambda}$
3. $\lim_{n\to\infty} Cov(T_1, T_n) = 0$

Simulation Conditions: c=2, d=1/2, λ=0.4



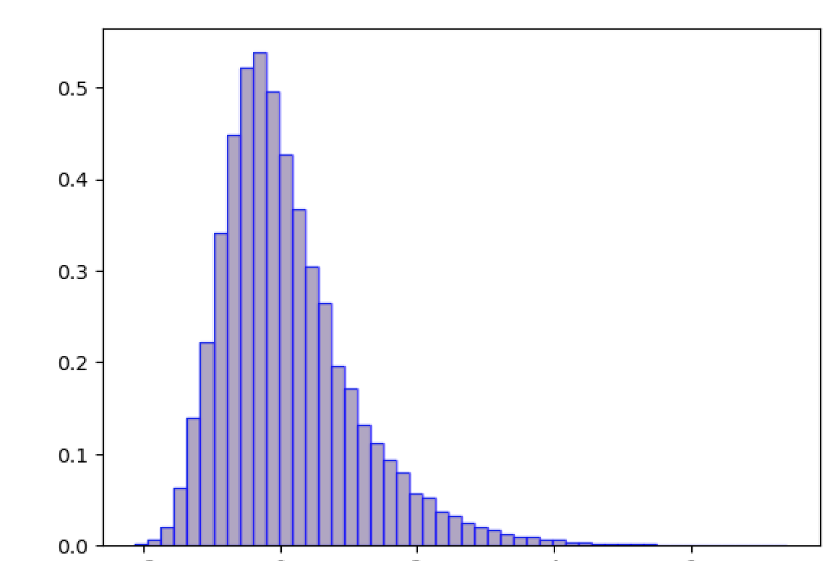
The variance of time that it takes n cars to get through the light increases linearly with n



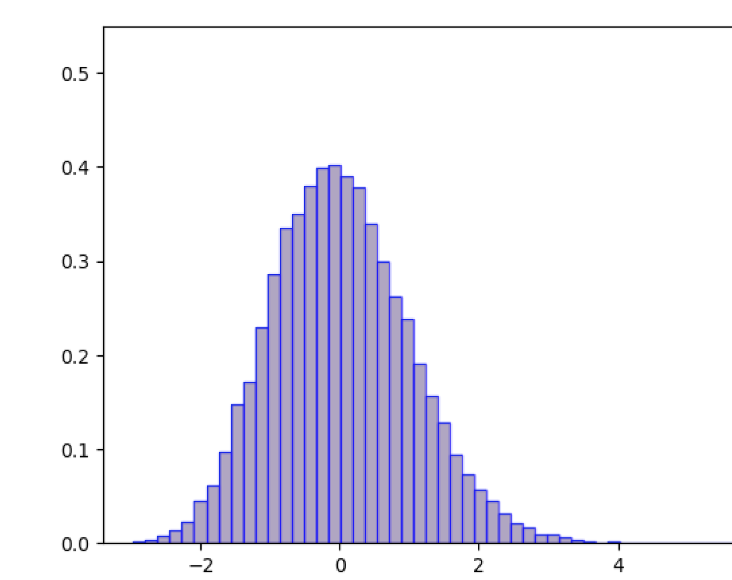
Results (continued)

Even though the conditions of the Central Limit Theorem are violated due to dependence of the car wait times, we used the asymptotic uncorrelation and convergence in distribution to suggest that an appropriately scaled sum of the car wait times may have an approximately standard normal distribution asymptotically.

Sum of 50 Car Times, 20000 samples



Sum of 1000 Car Times, 20000 Samples



The result of the Probability Plot Correlation Coefficient Test for normality at α=0.05 suggested nonnormality at 50 cars but normality with 1000 cars.

Conclusions and Future Directions

- When there are a sufficiently large number of cars waiting to make the left turn, the time it will take all of the cars to get through the light may be able to be treated as a normal random variable, but for more realistic numbers of cars (<50), the sum of their leaving times is not normal.
- Additional work is needed to prove asymptotic normality
- Additional future directions include incorporating designated left turn time in the model and allowing for multiple left turn lanes.

References

1. Filliben, J. J. (1975). The probability plot correlation coefficient test for normality. *Technometrics, 17*(1), 111-117. doi:10.1080/00401706.1975.10489279