A New Upper Bound for the Site Percolation Threshold of the Square Lattice

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Introduction

Percolation theory provides insight into the behavior of flow through systems. Applications include modeling the spread of disease in a population, contagion in the bank failures, connectivity of social media users, flow of fluid through a porous media, and much more.

Objective

A particularly important percolation model is the square lattice. The site percolation threshold is the smallest probability p such that when each vertex (*site*) is retained with probability greater than p, there exists an infinite connected component with probability 1. This important threshold value is unknown for the square lattice. We present a series of improvements to the upper bound for the site percolation threshold obtained through the substitution method.

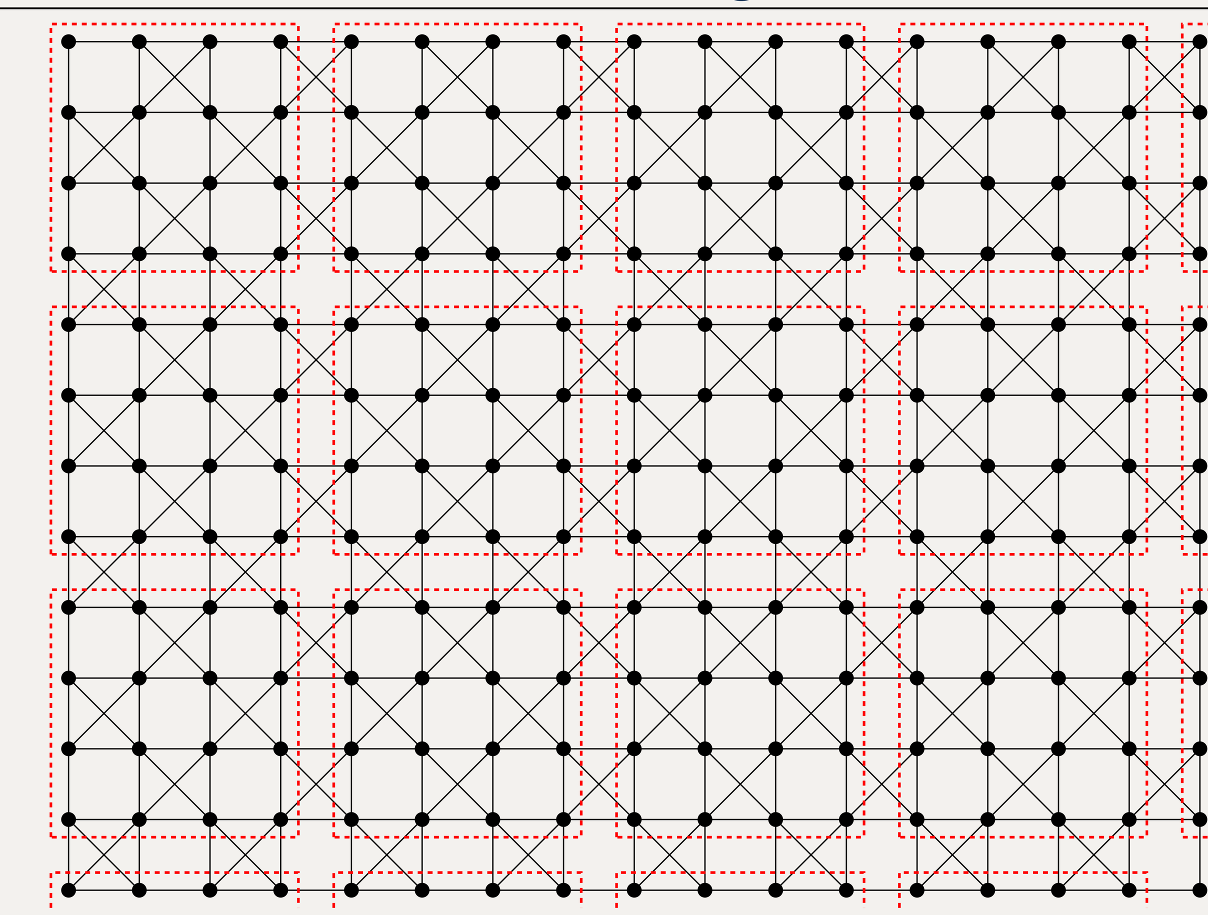
Methods

Improvements to the substitution method algorithm enabled the decreases in the upper bound. The substitution method leverages a direct comparison between a lattice with known site percolation threshold and one that we wish to investigate. Specific *substitution regions* that partition the vertices of the lattice are created and we create a stochastic ordering of probability measures to determine the bounds.

Self-Matching Lattice

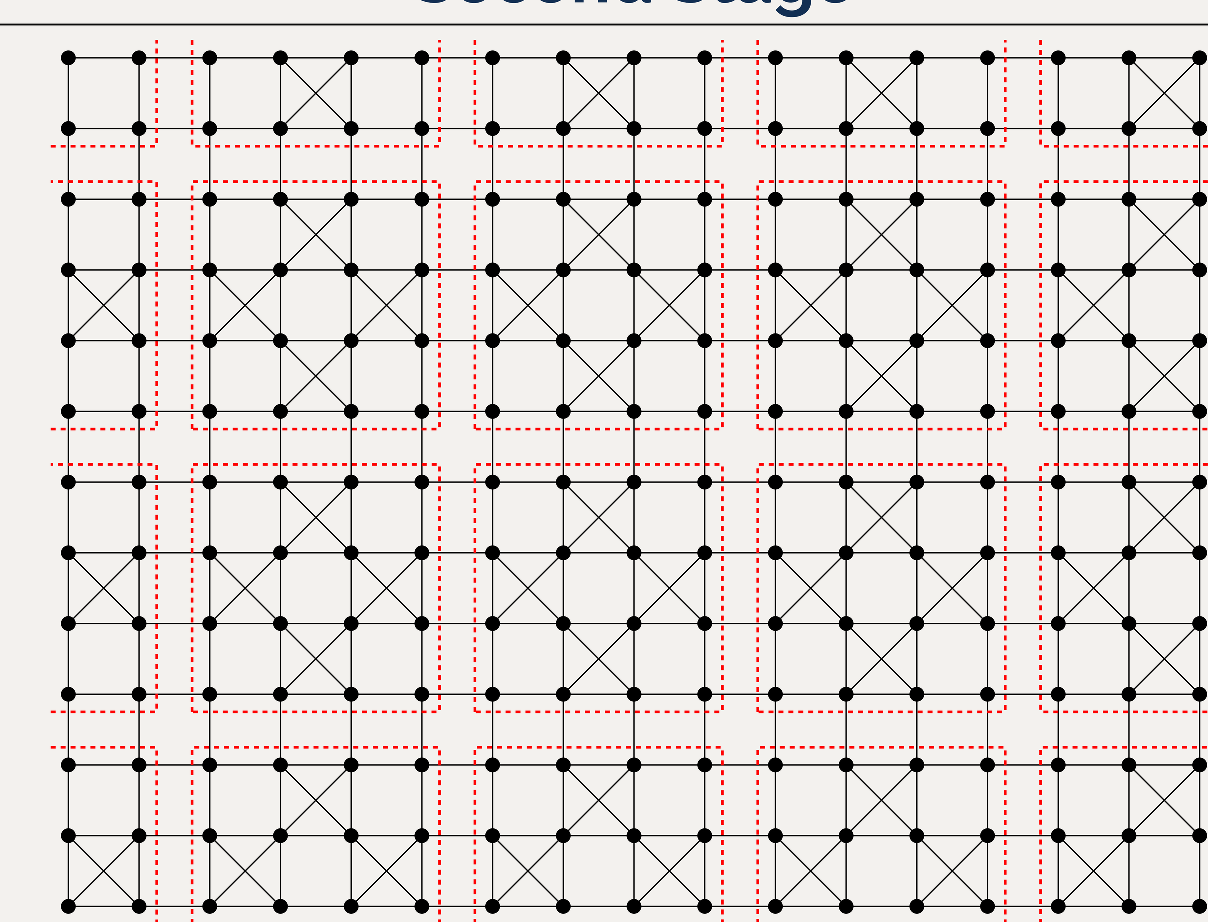
This is a self-matching lattice since replacing closed-packed squares with empty squares and vice versa results in an isomorphic lattice.

First Stage



The areas within the red dashes are the substitution regions. These regions have 32 boundary vertices that feed into 16 internal vertices. There are thus $2^{16} = 65536$ configurations of vertices and $Bell(32)$ partitions of the boundary vertices.

Second Stage



The second stage has the same internal configuration as the first stage. The computation resulted in the upper bound of 0.662869.

History of Improvements

Consensus Simulation Estimate $0.592746.$

Kesten (1981): $\frac{1}{2} \leq p_c^{site}(Square).$

Higuchi (1982): $\frac{1}{2} < p_c^{site}(Square).$

Toth (1985): $0.503478 \leq p_c^{site}(Square).$

Zuev (1988): $0.509535 \leq p_c^{site}(Square) \leq 0.681890.$

Wierman (1995): $p_c^{site}(Square) \leq 0.679492.$

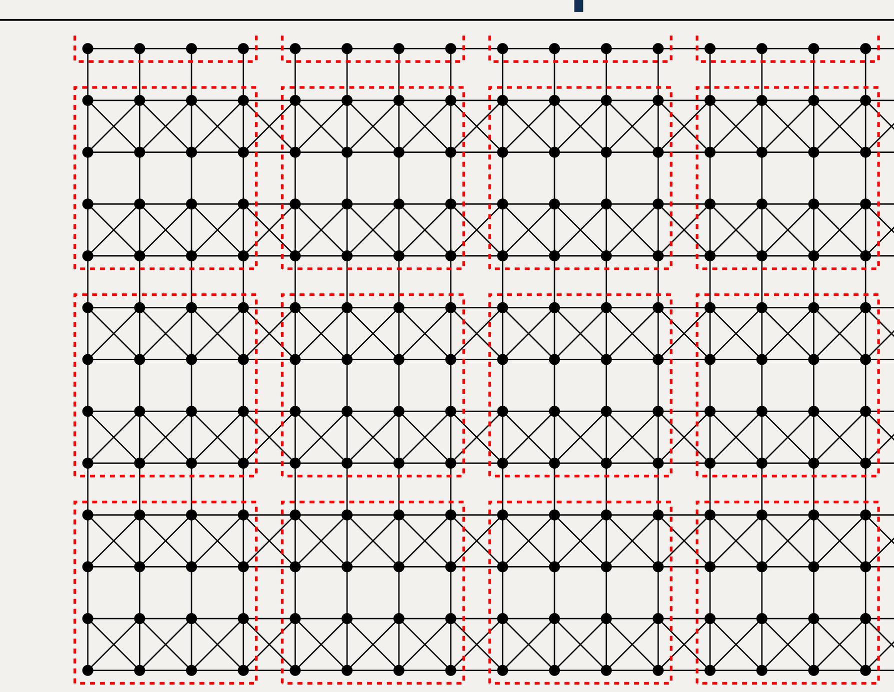
van den Berg and Ermakov (1996): $0.556000 < p_c^{site}(Square).$

Wierman and Oberly (1/2022): $p_c^{site}(Square) < 0.669789.$

Wierman and Oberly (2/2022): $p_c^{site}(Square) < 0.666894.$

Wierman and Oberly (2/2024): $p_c^{site}(Square) < 0.662869.$

Current Computation



Designer matching lattice with a 3x3 substitution region. This is the first stage. The second stage will consist just of the closed-packed squares between regions.

Conclusion

After a long period of no progress, we have provided a series of sizable improvements. The improvements to the substitution method algorithm are immediately applicable to other lattices and thus we believe we can continue to improve on bounds for percolation thresholds.

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