Optimizing Polyhedral Passages via Gradient Methods

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A 400 Year Old Geometry Question

Surprisingly, given two equally large cubes, one can pass the first through a hole strictly inside the second. This property was first shown by Prince Rupert of the Rhine in the 17th century. This isn't a unique property of cubes, rather, it is holds for many polyhedrons, motivating the question:

Does every Polyhedron have a Polyhedral Passage?

In 1983, mathematicians [1] began to approach this problem algorithmically, seeking to find the largest (rescaled) copy of a given polyhedron that can pass strictly through itself. As recently as 2021, probabilistic techniques were tried and conjectured a counterexample: the rhombicosidodecahedron [2]. We aim to answer this question via more efficient algorithm for computing the largest rescaling for any given polyhedron.

Methodology: Optimizing Polyhedral Passages

Rotations and Translations as Variables x

Two initial copies of a shape, they need to be moved and spun in some way for one to pass strictly through the other. We can describe all possible movements as rotating the shapes in 3D via rotations around the X, Y, and Z axes (using 2D rotation matrices), and translating in the X and Y directions. In our code, we fix that the passage will happen in the Z direction. Hence the Z coordinate does not need to be translated. This leaves eight parameters describing any possible passage, which we collect in a vector "x".

The Largest Possible Rescaling: μ(x)

For given rotations and translations x, we define "μ(x)" to be the function that finds the largest rescaling of the inner shape (e.g., the cube making the passage) that strict fits through the other. We compute this by it iterating edge by edge over the outer shape to see how much the inner shape could be rescaled in while fitting within that edge. Finally, it returns the smallest rescaling value it found over all of these edges, giving us the largest value for which this translated and rotated inner shape could be dilated by and still fit within the other rotated outer shape.

Maximizing μ(x) via Gradient Ascent

We make use of gradient ascent optimize our "μ(x)“. From a given initialization of the translation/rotation parameters, we numerically calculate the gradient in each direction and then move an amount "α“ in that direction. We repeat this process many times until a local maximum of μ(x) is reached (improvements with backtracking and stopping criterions were also used):

$$x_{k+1} = x_k + \alpha_k \nabla \mu(x_k)$$

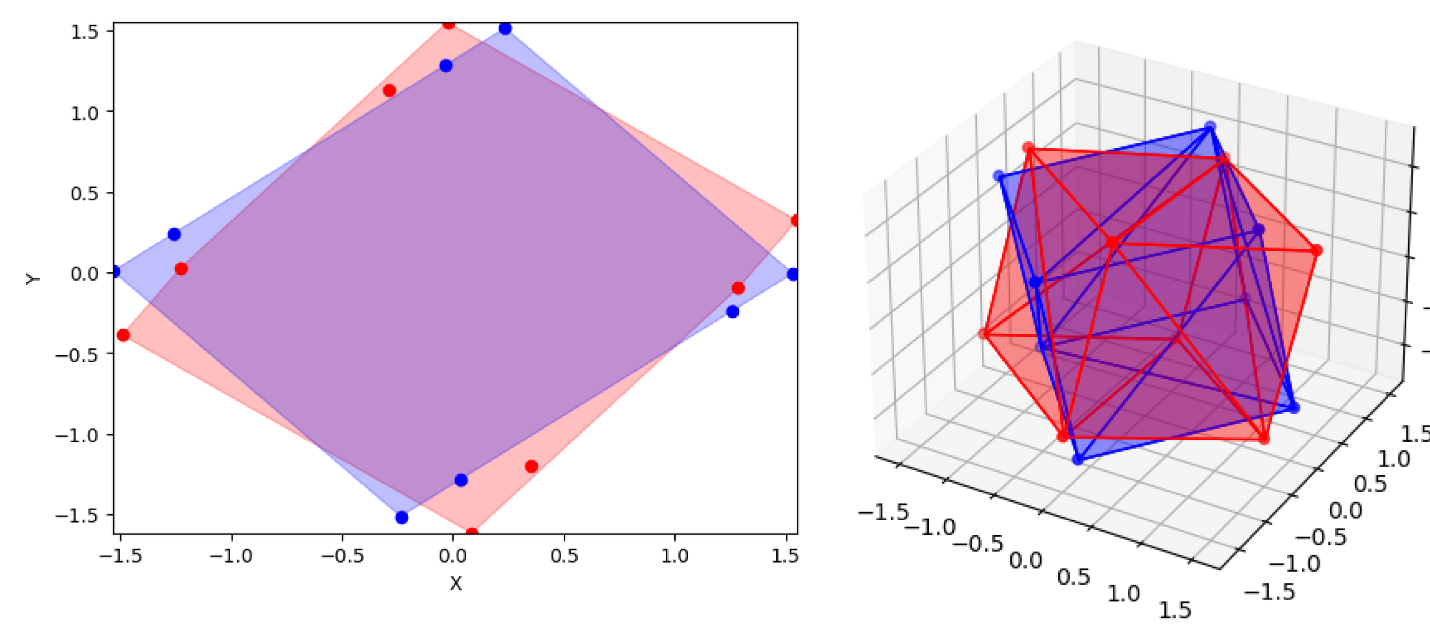
2D and 3D Plots of Passages

Using Matplotlib, we can visualize the polyhedral passage given by each parameter selection *x*. The 2D plots the projected “shadow” of each shape and the 3D plots show the actual shapes.

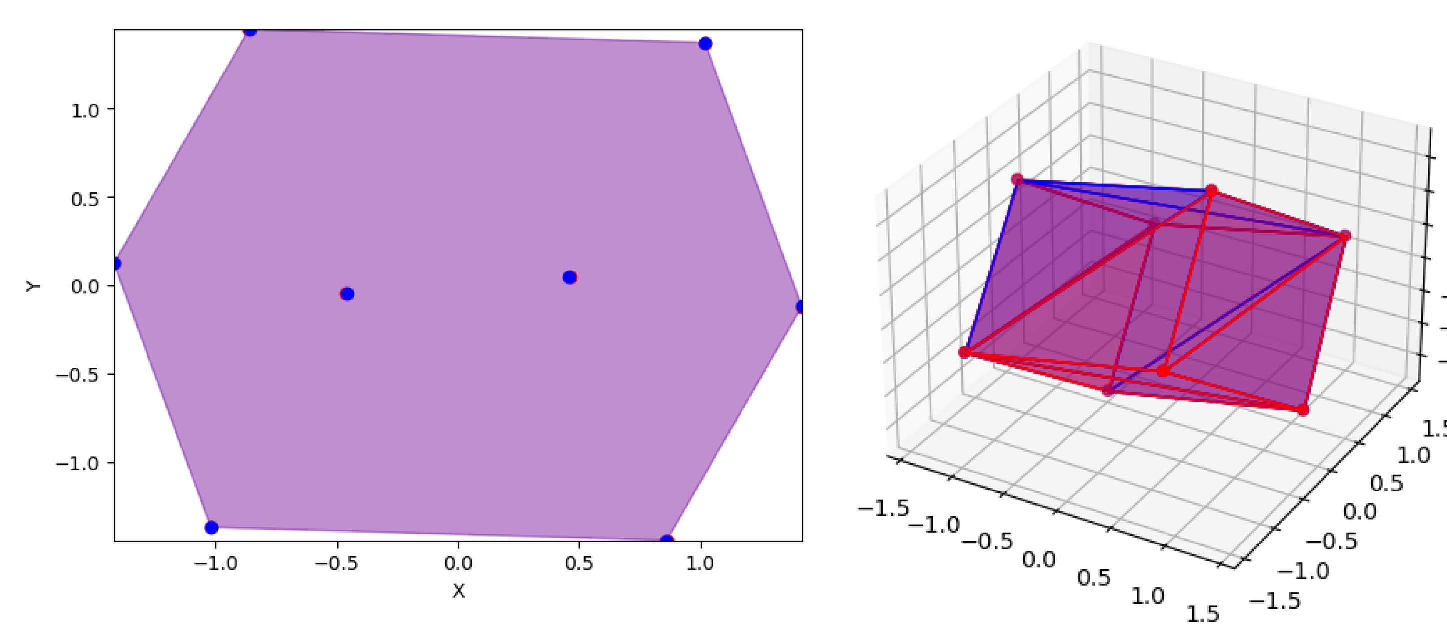
Our goal is to have the red cube pass strictly within the blue cube. The 2D plots show the shadows of both the red and blue cubes by dropping their Z coordinates. In theory, we want to pass the red cube through a hole in the blue cube, with our rescaling value telling us how much larger we can make the red cube such that it still passes strictly through the hole in the blue cube.

Below we show failed, marginal and successful passages.

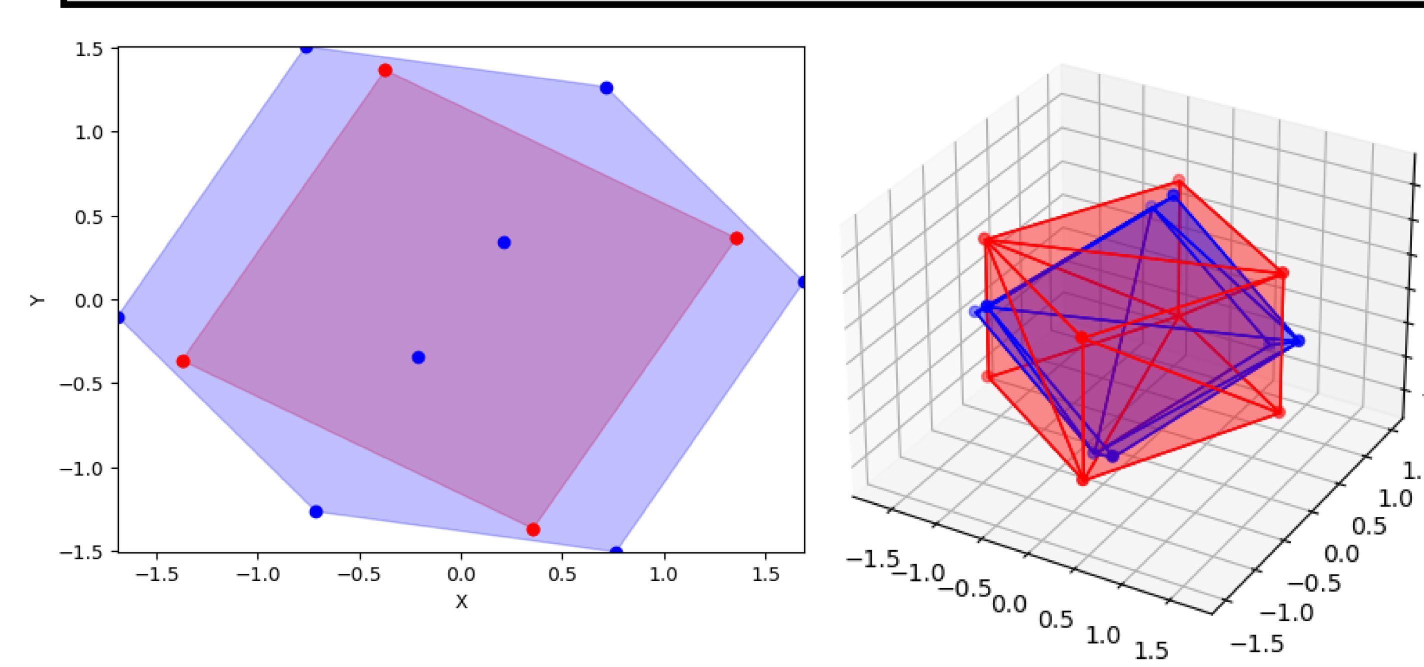
A Failed Passage: μ(x) < 1



A Marginal Passage: μ(x) = 1



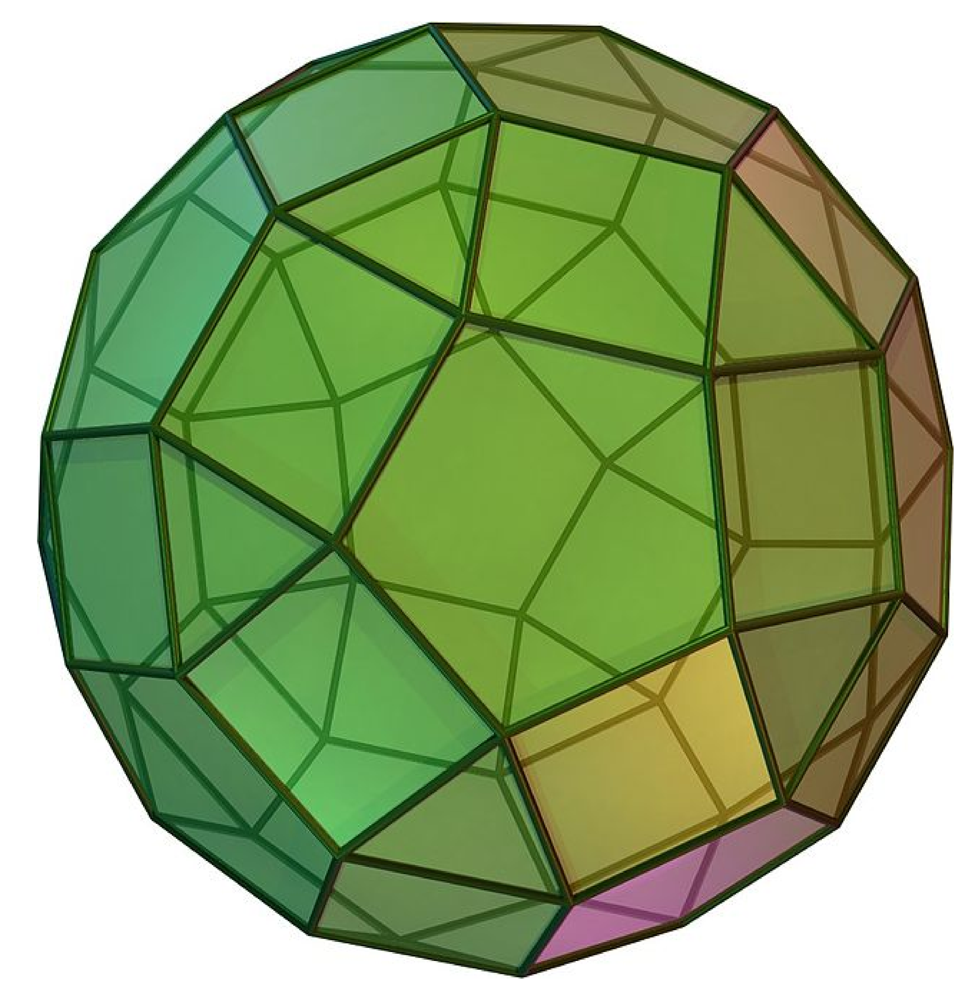
Successful Passage: μ(x) > 1



Improvements on Archimedean Solids

Solid	Old Best	New Best
Rhombicosidodecahedron	< 1	1.00135024
Tetrahedron	1.014473	1.01459894
Icosahedron	1.010805	1.01081856
Truncated Icosahedron	1.001955	1.00196198
Truncated Cuboctahedron	1.006563	1.00658547

Our Newly Proved Archimedean Solid



The Rhombicosidodecahedron has 62 faces: 20 triangular, 30 square, and 12 pentagonal, with 60 vertices and 120 edges. While it was believed by many to not follow Rupert's Property, we found that we can make it 0.1% larger and still find a passage. It also gives us hope that every Archimedean Solid follows Rupert's Property; while we are still missing a lower bound for two of the 13 Archimedean Solids, we still believe that either through higher precision or different methods these two shapes can be reached as well.

Future Directions

This is the conclusion of our work with polyhedral passages. We have implemented our nonsmooth optimization algorithm on the three major classes of polyhedron within the field (Archimedean, Johnson, and Catalan), and have discovered many new lower bounds for the maximum rescaling of these polyhedra, even disproving potential counterexamples of shapes thought to not have Rupert's Property. While our results do not lead us to any breakthroughs in engineering applications, the use of gradient ascent to attack nonsmooth optimization in many dimensions is a novel process within this subfield that has pushed the boundaries on Rupert's Property. We are thrilled with everything we have accomplished and are excited to see the bounds of all polyhedron continue to increase.

Citations

[1] Steininger, Jakob, and Sergey Yurkevich. "An algorithmic approach to Rupert's problem." arXiv:2103.01808 (2021).
[2] B.Chazelle. "The Polygon Containment Problem" Advances in Computing Research I, pages1–33,1983.