Finding a Compositional Square Root of Sine

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Abstract

We consider the following type of problem: Given a function $g: A \to A$, find a function $f: A \to A$ such that $g = f \circ f$. We are especially interested in the case $sin: \mathbf{R} \to \mathbf{R}$, but consider the problem more broadly with results for other functions g defined on other sets A.

Introduction

The general question is this: Given a function g defined on a set A, $g: A \to A$, find a function $f: A \to A$ such that $g = f \circ f$.

When $g: A \to A$ is repeatedly applied to itself, the results are called *iterates*. We use superscripts to denote iteration; for a positive integer n, we have

$$g^n = \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ terms}}$$

where there are n terms in the composition. g^0 is the identity function, i.e., $g^0(x) = x$. In case $g: A \to A$ is a bijection, then g has an inverse and negative exponents are meaningful.

A function f with the property that $f^2 = g$ is often called a *half iterate* of g, but we prefer to say that f is a *compositional square root* of g. While our objective is to find a compositional square root of sine, a few examples help us frame the general issue.

Examples

1. We begin with an easy example. Let $g: \mathbf{R} \to \mathbf{R}$ by $g(x) = 2x$. Then clearly $f(x) = \sqrt{2}x$ has the desired property. However, if we consider the same function g but restrict to the rational numbers, $g: \mathbf{Q} \to \mathbf{Q}$, it is not clear if there is a solution (there is).

2. Similarly, let $g(x) = -x$. If the set in question is the set of complex numbers, $g: \mathbf{C} \to \mathbf{C}$, then there is a simple solution: $f(x) = ix$. However, if we restrict g (and f) to the reals, it is not immediately apparent that there is a solution (there is), but there is no continuous solution.

Note: We had various interesting *examples* (and *examples revisited*) where we use the weaving method to construct the compositional square root of different types of functions by analyzing their orbits (i.e., connected components). Due to limited space, we will show the simplest example(s) here only to illustrate the problem and demonstrate how the weaving method works.

Functional Graphs

To understand when a function $g: A \to A$ has a compositional square root, it is useful to consider the following directed graph, G_g. The vertices of G_g are the elements of A. In G_g there is a directed edge $x \to y$ exactly when $g(x) = y$.

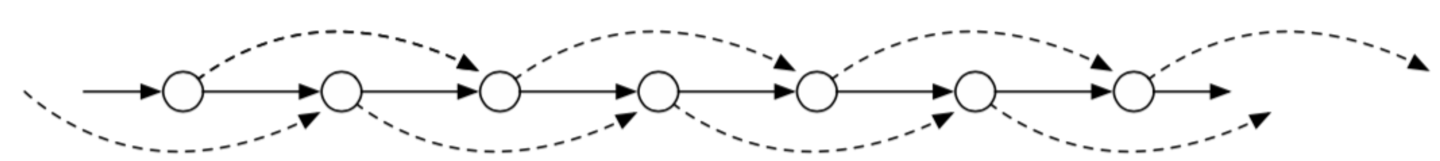


Figure 1: The relation between the directed graph G_f (solid lines) and the graph G_g (dashed lines), where $g = f^2$.

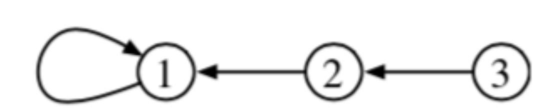


Figure 2: The graph of a function that does not have a compositional square root.

The Weaving Method

In this construction we weave together isomorphic components of G_g to give components of G_f for a new function $f: A \to A$ with $f^2 = g$. Let H_1, H_2 be connected components of G_g and let $\phi: V(H_1) \to V(H_2)$ be a bijection between the vertex sets of H_1 and H_2 that established the isomorphism of these graphs.

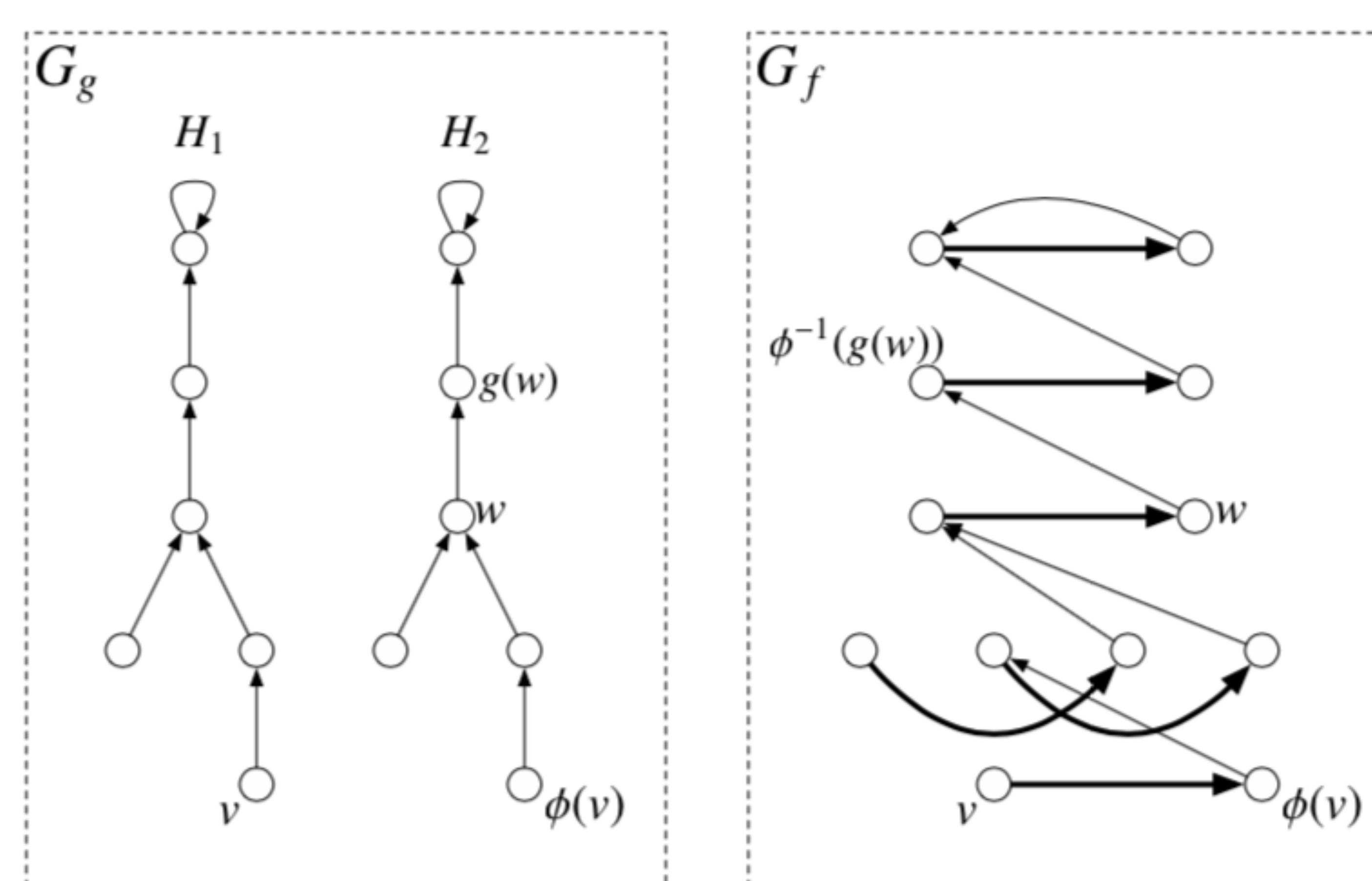


Figure 3: Illustration of the weaving method. In a vertex v of H_1 has an edge pointing to $\phi(v)$ (shown as a stick arrow) and a vertex w of H_2 has an edge pointing to $\phi^{-1}(g(w))$ (shown as a thin arrow).

Example Revisited

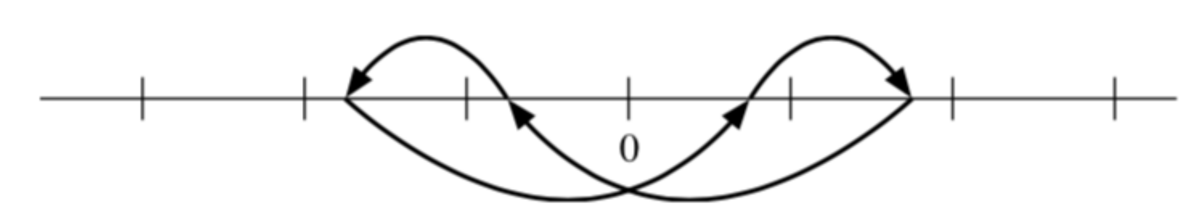
Let $g: \mathbf{Q} \to \mathbf{Q}$ by $g(x) = 2x$. Define bijection ϕ as follows:

$$\phi(x) = \begin{cases} x+1 & \text{if } \lceil x \rceil \text{ is odd, and} \\ x-1 & \text{otherwise.} \end{cases}$$

With this in place, we have the following compositional square root of g:

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x+1 & \text{if } x > 0 \text{ and } \lceil x \rceil \text{ is odd,} \\ -(x-1) & \text{if } x > 0 \text{ and } \lceil x \rceil \text{ is even,} \\ x-1 & \text{if } x < 0 \text{ and } \lfloor x \rfloor \text{ is odd, and} \\ -x-1 & \text{if } x < 0 \text{ and } \lfloor x \rfloor \text{ is even.} \end{cases}$$

The function f is illustrated below (Figure 4):

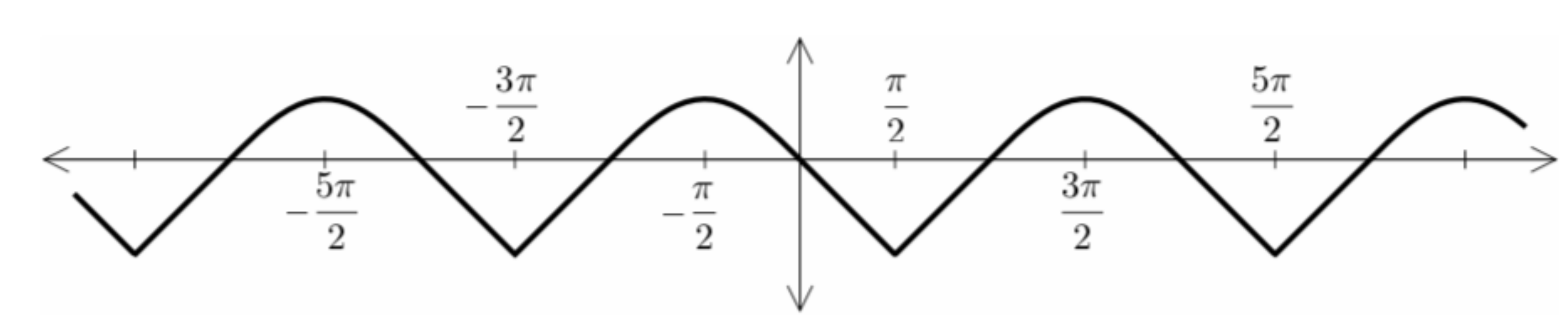


A Compositional Square Root of Sine

We constructed a continuous compositional square root of sine using the weaving method. Let $\phi: (1, \pi/2\,] \to [-\pi/2, -1)$ be defined by $\phi(x) = -x$. We weave orbits together to give the following function:

$$f(x) = \begin{cases} -x & \text{if } x \geq 0 \text{ and} \\ -\sin x & \text{if } x < 0. \end{cases}$$

Furthermore, we can extend this function to the entire real line to form a continuous compositional square root of sine defined for all real numbers, as illustrated below:



Half Derivatives

A half derivative is an operator that when applied twice is equivalent to the derivative operator. We discovered that there are infinitely many nonlinear (and no linear) half derivative operators from $\mathscr{C}^\infty$ to $\mathscr{C}^\infty$, where $\mathscr{C}^\infty$ is the set of functions $f: \mathbf{R} \to \mathbf{R}$ with derivatives of all orders, and we can construct them using the weaving method as well. Due to limited space, we will omit the interesting discoveries and details we have on half derivatives.